

# A VANISHING THEOREM OF KOLLÁR-OHSAWA TYPE

SHIN-ICHI MATSUMURA

**ABSTRACT.** For proper surjective holomorphic maps from Kähler manifolds to analytic spaces, we give a decomposition theorem for the cohomology groups of the canonical bundle twisted by Nakano semi-positive vector bundles by means of the higher direct image sheaves, by using the theory of harmonic integrals developed by Takegoshi. As an application, we prove a vanishing theorem of Kollár-Ohsawa type by combining the  $L^2$ -method for the  $\bar{\partial}$ -equation.

## 1. INTRODUCTION

The study of the direct image sheaves of adjoint bundles is one of important subjects in algebraic geometry and the theory of several complex variables. The main purpose of this paper is to generalize results related to this subject in [Kol86a], [Kol86b], [Ohs84], [Tak95]. In this paper, for a proper surjective holomorphic map  $f: X \rightarrow Y$  from a Kähler manifold  $X$  to an analytic space  $Y$ , we consider the higher direct image sheaves  $R^q f_*(K_X \otimes E)$  of the canonical bundle  $K_X$  twisted by a vector bundle  $E$  and the Leray spectral sequence

$$H^p(Y, R^q f_*(K_X \otimes E)) \implies H^{p+q}(X, K_X \otimes E).$$

In his paper [Tak95], Takegoshi proved the degeneration of the Leray spectral sequence at  $E_2$ -term and an injectivity theorem for the case  $p = 0$  when  $E$  admits a (hermitian) metric with semi-positive curvature in the sense of Nakano. (See [EV92], [Kol86a], [Kol86b], [Sko78] for related topics, and see [Fn15], [Mat15] for recent developments.) In order to generalize these results, we study the Leray spectral sequence in detail, by using the theory of harmonic integrals developed by Takegoshi. As a result, we obtain a decomposition theorem and an injectivity theorem for the case  $p > 0$  (Theorem 1.1 and Corollary 1.2). Theorem 1.1 is stronger than the degeneration at  $E_2$ -term, and Corollary 1.2 is a generalization of the corollary of the main result in [Kol86b]. Moreover, as an application, we prove a generalization of the vanishing theorem of Kollár-Ohsawa type (Theorem 1.3), which gives an affirmative answer for [Fn15, Conjecture 2.25]. The following theorem can be seen as a weak form of the decomposition theorem.

**Theorem 1.1.** *Let  $f: X \rightarrow Y$  be a proper surjective holomorphic map from a Kähler manifold  $X$  to an analytic space  $Y$ , and  $E$  be a vector bundle on  $X$  admitting a (hermitian)*

---

*Key words and phrases.* Decomposition theorems, Vanishing theorems, Higher direct images, Kähler deformations, the Leray spectral sequence, the theory of harmonic integrals,  $L^2$ -methods for  $\bar{\partial}$ -equations.

Classification AMS 2010:Primary 32L20, Secondary 14C30, 58A14.

metric with semi-positive curvature in the sense of Nakano. Then, for integers  $p, q \geq 0$ , there exists a natural homomorphism

$$\varphi_{p,q}: H^p(Y, R^q f_*(K_X \otimes E)) \rightarrow H^{p+q}(X, K_X \otimes E)$$

with the following properties:

- The homomorphism  $\varphi_{p,q}$  is injective.
- $\text{Im } \varphi_{p,q} \cap \text{Im } \varphi_{p',q'} = \{0\}$  when  $p \neq p'$  and  $p + q = p' + q'$ .

Here  $K_X$  denotes the canonical bundle of  $X$ ,  $R^q f_*(\bullet)$  denotes the  $q$ -th higher direct image sheaf, and  $\text{Im } \varphi_{p,q}$  denotes the image of  $\varphi_{p,q}$ .

When  $X$  is a compact Kähler manifold, we obtain a decomposition theorem for  $H^\ell(X, K_X \otimes E)$  (the first conclusion of Corollary 1.2). Moreover, we also obtain an injectivity theorem for the multiplication map induced by (holomorphic) sections of semi-positive line bundles (the latter conclusion of Corollary 1.2).

**Corollary 1.2.** *Under the same situation as in Theorem 1.1, we further assume that  $X$  is a compact Kähler manifold. Then we have the following direct sum decomposition:*

$$H^\ell(X, K_X \otimes E) = \bigoplus_{p+q=\ell} \text{Im } \varphi_{p,q}.$$

Moreover, for a line bundle  $F$  on  $X$  admitting a metric with semi-positive curvature and a (non-zero) section  $s$  of  $F^m$  ( $m > 0$ ), the multiplication map induced by the tensor product with  $s$

$$\Phi_s: H^p(Y, R^q f_*(K_X \otimes E \otimes F)) \xrightarrow{\otimes s} H^p(Y, R^q f_*(K_X \otimes E \otimes F^{m+1}))$$

is injective for any  $p, q \geq 0$ .

Note that the injectivity theorem for the case  $p = 0$  follows from [Tak95, Injectivity Theorem III]. In [Fs15], the above decomposition was generalized to a decomposition theorem in the derived category. By applying the proof of Theorem 1.1 (in particular, the construction of  $\varphi_{p,q}$ ) and the  $L^2$ -method for the  $\bar{\partial}$ -equation, we can prove a vanishing theorem for the higher cohomology groups of  $R^q f_*(K_X \otimes E)$  if  $E$  admits a metric  $h$  satisfying  $\sqrt{-1}\Theta_h(E) \geq f^*\sigma \otimes \text{id}_E$  for some Kähler form  $\sigma$  on  $Y$  in the sense of Nakano (see Section 4 for the definition of Kähler forms on analytic spaces). When  $X$  is compact and  $q = 0$ , Ohsawa first proved such a vanishing theorem in [Ohs84], by giving the elegant method to solve the  $\bar{\partial}$ -equation on complete Kähler manifolds and inductively constructing solutions of the  $\bar{\partial}$ -equation. (See Remark 4.4 for a comparison between our proof and Ohsawa's proof.) In [Kol86a], Kollár proved the same conclusion when  $X$  is a smooth projective variety and  $E$  is the pull-back of an ample line bundle. Therefore the following theorem can be seen as a generalization of Ohsawa's result to higher direct images and of Kollár's result to the complex analytic setting.

**Theorem 1.3.** *Let  $f: X \rightarrow Y$  be a proper surjective holomorphic map from a weakly pseudoconvex Kähler manifold  $X$  to an analytic space  $Y$ , and  $E$  be a vector bundle on  $X$ .*

If  $E$  admits a metric whose curvature is larger than or equal to the pull-back of a Kähler form on  $Y$  in the sense of Nakano, then we have

$$H^p(Y, R^q f_*(K_X \otimes E)) = 0 \quad \text{for any } p > 0 \text{ and } q \geq 0.$$

In particular, we have the natural isomorphism  $H^0(Y, R^q f_*(K_X \otimes E)) \cong H^q(X, K_X \otimes E)$ .

This paper is organized as follows: In Section 2, we will briefly recall results on the theory of harmonic integrals developed in [Tak95]. We will prove Theorem 1.1 and Corollary 1.2 in Section 3, and prove Theorem 1.3 in Section 4.

**Acknowledgement.** The author would like to thank Professors Osamu Fujino and Taro Fujisawa for reading the draft and giving useful comments. He is supported by the Grant-in-Aid for Young Scientists (B) #25800051 from JSPS.

## 2. PRELIMINARIES

In this section, we fix the notation used in this paper and summarize the theory of harmonic integrals developed in [Tak95].

Throughout this paper, let  $f: X \rightarrow Y$  be a proper surjective holomorphic map from a Kähler manifold  $X$  with a Kähler form  $\omega$  to an analytic space  $Y$ , and  $E$  be a vector bundle on  $X$  with a hermitian metric  $h$ . Let  $n$  be the dimension of  $X$ . For a Stein open set  $U$  in  $Y$ , we take an exhaustive smooth plurisubharmonic function  $\Psi_U$  on  $U$ , and put  $\Phi_U := f^* \Psi_U$ , which is also exhaustive and plurisubharmonic. Following Takegoshi's result in [Tak95], we define the space of harmonic forms on  $X_U := f^{-1}(U)$  by

$$\mathcal{H}^{n,k}(X_U, E, \Phi_U) = \{u \in C_\infty^{n,k}(X_U, E) \mid \bar{\partial}u = 0, \bar{\partial}^*u = 0, (\bar{\partial}\Phi_U)^*u = 0 \text{ on } X_U\},$$

where  $\bar{\partial}^*$  (resp.  $(\bar{\partial}\Phi_U)^*$ ) is the adjoint operator of  $\bar{\partial}$  (resp. the wedge product  $\bar{\partial}\Phi_U \wedge \bullet$ ). This space can actually be shown to be independent of the choice of exhaustive plurisubharmonic functions if the curvature of  $h$  is semi-positive (see [Tak95, Theorem 4.3 (ii)]). The following theorem plays a central role in the proof of Theorem 1.1.

**Theorem 2.1.** ([Tak95, Theorem 4.3, 5.2]). *Under the same notation as above, we assume that the curvature of  $h$  is semi-positive in the sense of Nakano. Then we have the following:*  
(1) *The natural quotient map from the space of smooth  $\bar{\partial}$ -closed  $E$ -valued  $(n, k)$ -forms to the (Dolbeault) cohomology group induces the isomorphism*

$$\mathcal{H}^{n,k}(X_U, E, \Phi_U) \xrightarrow{\cong} H^k(X_U, K_X \otimes E).$$

(2) *For Stein open sets  $U_1, U_2$  in  $Y$  such that  $U_2 \subset U_1$ , the restriction map*

$$\mathcal{H}^{n,k}(X_{U_1}, E, \Phi_{U_1}) \longrightarrow \mathcal{H}^{n,k}(X_{U_2}, E, \Phi_{U_2})$$

is well-defined, and further it satisfies the following commutative diagram:

$$\begin{array}{ccc} \mathcal{H}^{n,k}(X_{U_1}, E, \Phi_{U_1}) & \xrightarrow{\cong} & H^k(X_{U_1}, K_X \otimes E) \\ \downarrow & & \downarrow \\ \mathcal{H}^{n,k}(X_{U_2}, E, \Phi_{U_2}) & \xrightarrow{\cong} & H^k(X_{U_2}, K_X \otimes E). \end{array}$$

In this paper, we denote by  $\mathcal{H}_U$ , the inverse map of the natural quotient map in the above theorem, namely

$$\mathcal{H}_U: H^k(X_U, K_X \otimes E) \xrightarrow{\cong} \mathcal{H}^{n,k}(X_U, E, \Phi_U).$$

Further we often omit the subscript. To avoid confusion in the proof of Theorem 1.1, we should clearly distinguish between the (Dolbeault) cohomology class and the  $\bar{\partial}$ -closed  $E$ -valued form. For this reason, we denote the equality in  $H^k(X_U, K_X \otimes E)$  by  $\equiv$ , and the cohomology class determined by a  $\bar{\partial}$ -closed  $E$ -valued form  $\bullet$  by  $[\bullet]$ . We remark that we have  $\alpha \equiv [\mathcal{H}_U(\alpha)]$  for an arbitrary cohomology class  $\alpha \in H^k(X_U, K_X \otimes E)$ .

In the proof of Theorem 1.3, we consider the  $L^2$ -space of  $E$ -valued  $(n, k)$ -forms on  $X$  with respect to  $h$  and  $\omega$  defined by

$$L_{(2)}^{n,k}(X, E)_{h,\omega} := \{u \mid u \text{ is an } E\text{-valued } (n, k)\text{-form on } X \text{ with } \|u\|_{h,\omega} < \infty\}.$$

Here  $\|u\|_{h,\omega}$  denotes the  $L^2$ -norm defined by

$$\|u\|_{h,\omega}^2 := \langle\langle u, u \rangle\rangle_{h,\omega} := \int_X \langle u, u \rangle_{h,\omega} dV_\omega,$$

where  $\langle u, u \rangle_{h,\omega}$  is the point-wise inner product with respect to  $h, \omega$  and  $dV_\omega$  is the volume form defined by  $dV_\omega := \omega^n/n!$ . By the Bochner-Kodaira-Nakano identity, we obtain

$$\langle\langle \sqrt{-1}\Theta_h(E)\Lambda_\omega v, v \rangle\rangle \leq \|\bar{\partial}v\|^2 + \|\bar{\partial}^*v\|^2$$

for compactly supported and smooth  $E$ -valued  $(n, k)$ -form  $v$ . Here  $\Lambda_\omega$  is the (point-wise) adjoint operator of the wedge product  $\omega \wedge \bullet$  with respect to  $\langle \bullet, \bullet \rangle_{h,\omega}$ .

### 3. DECOMPOSITION BY COHOMOLOGY GROUPS OF HIGHER DIRECT IMAGES

The purpose of this section is to prove Theorem 1.1 and Corollary 1.2. We will give the construction of the homomorphism  $\varphi_{p,q}$  in subsection 3.2, by using the harmonic forms representing cohomology classes. Further we will show that  $\varphi_{p,q}$  is injective in subsection 3.3 and it gives a direct sum in subsection 3.4.

**3.1. Set up.** Throughout this section, we freely use the notation in the statement of the following theorem.

**Theorem 3.1** (=Theorem 1.1). *Let  $f: X \rightarrow Y$  be a proper surjective holomorphic map from a Kähler manifold  $X$  to an analytic space  $Y$ , and  $E$  be a vector bundle on  $X$  admitting*

(hermitian) metric with semi-positive curvature in the sense of Nakano. Then, for integers  $p, q \geq 0$ , there exists a natural homomorphism

$$\varphi_{p,q}: H^p(Y, R^q f_*(K_X \otimes E)) \rightarrow H^{p+q}(X, K_X \otimes E)$$

with the following properties:

- The homomorphism  $\varphi_{p,q}$  is injective.
- $\text{Im } \varphi_{p,q} \cap \text{Im } \varphi_{p',q'} = \{0\}$  when  $p \neq p'$  and  $p + q = p' + q'$ .

We fix a Kähler form  $\omega$  on  $X$  and a metric  $h$  on  $E$  whose curvature is semi-positive in the sense of Nakano. By the assumption of the curvature of  $h$ , we can represent a cohomology class  $\alpha$  by the associated harmonic form  $\mathcal{H}(\alpha)$  on  $f^{-1}(U)$  for a Stein open set  $U$  in  $Y$ .

**3.2. Construction of  $\varphi_{p,q}$ .** First we take a Stein open cover  $\mathcal{U} := \{U_i\}_{i \in I}$  of  $Y$ , and consider the standard isomorphism

$$H^p(Y, R^q f_*(K_X \otimes E)) \cong \check{H}^p(\mathcal{U}, R^q f_*(K_X \otimes E)),$$

where the right hand side is the Čech cohomology group calculated by  $\mathcal{U}$ . Since the open set  $U_{i_0 \dots i_p} := U_{i_0} \cap \dots \cap U_{i_p}$  is also Stein, we have the natural isomorphism

$$H^0(U_{i_0 \dots i_p}, R^q f_*(K_X \otimes E)) \xrightarrow{\cong} H^q(f^{-1}(U_{i_0 \dots i_p}), K_X \otimes E).$$

By this isomorphism, we identify the  $p$ -cochains valued in  $R^q f_*(K_X \otimes E)$  with the  $p$ -cochains  $\{\alpha_{i_0 \dots i_p}\}_{i_0 \dots i_p}$  valued in the (Dolbeault) cohomology of  $E$ -valued  $(n, q)$ -forms. It is sufficient for the construction of  $\varphi_{p,q}$  to define  $\varphi_{p,q}(\{\alpha_{i_0 \dots i_p}\}_{i_0 \dots i_p})$  for a given  $p$ -cochain  $\{\alpha_{i_0 \dots i_p}\}_{i_0 \dots i_p}$  of the cohomology classes  $\alpha_{i_0 \dots i_p} \in H^q(f^{-1}(U_{i_0 \dots i_p}), K_X \otimes E)$  satisfying  $\delta(\{\alpha_{i_0 \dots i_p}\}_{i_0 \dots i_p}) \equiv 0$ . Here  $\equiv$  is the equality in the (Dolbeault) cohomology group and  $\delta$  is the coboundary operator defined by

$$\delta(\{\alpha_{i_0 \dots i_p}\}_{i_0 \dots i_p}) := \left\{ \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0 \dots \hat{i}_k \dots i_{p+1}}|_{f^{-1}(U_{i_0 \dots i_{p+1}})} \right\}_{i_0 \dots i_{p+1}}.$$

For simplicity, we put  $V_{i_0 \dots i_p} := f^{-1}(U_{i_0 \dots i_p})$ , and further we omit the notation of the restriction in the right hand side and the subscript, such as “ $i_0 \dots i_p$ ”.

For an arbitrary  $p$ -cocycle  $\{\alpha_{i_0 \dots i_p}\}$  of  $\alpha_{i_0 \dots i_p} \in H^q(V_{i_0 \dots i_p}, K_X \otimes E)$ , we will construct a  $\bar{\partial}$ -closed  $E$ -valued  $(n, p+q)$ -form on  $X$  and define  $\varphi_{p,q}(\{\alpha_{i_0 \dots i_p}\}) \in H^{p+q}(X, K_X \otimes E)$  by its cohomology class.

First we consider the case  $p = 0$ . In this case, by  $\delta(\{\alpha_{i_0}\}) \equiv 0$ , we have

$$\delta(\{\mathcal{H}(\alpha_{i_0})\}) = \mathcal{H}(\delta(\{\alpha_{i_0}\})) = 0.$$

Here we implicitly used the property (2) in Theorem 2.1. We remark that the above equality is the equality as the  $E$ -valued  $(n, q)$ -forms (not the cohomology classes). By the above equality, the family  $\{\mathcal{H}(\alpha_{i_0})\}$  is a 0-cocycle of the  $E$ -valued  $(n, q)$ -forms, and thus it determines the  $E$ -valued  $(n, q)$ -form globally defined on  $X$ . Then we define  $\varphi_{0,q}$  by  $\varphi_{0,q}(\{\alpha_{i_0}\}) := [\{\mathcal{H}(\alpha_{i_0})\}]$ , where  $[\bullet]$  denotes the cohomology class determined by a  $\bar{\partial}$ -closed  $E$ -valued form  $\bullet$ .

From now on, we consider the case  $p > 0$ . For a given  $p$ -cocycle  $\{\alpha_{i_0 \dots i_p}\}$ , by  $\delta(\{\alpha_{i_0 \dots i_p}\}) \equiv 0$ , we have

$$\delta(\{\mathcal{H}(\alpha_{i_0 \dots i_p})\}) = \mathcal{H}(\delta(\{\alpha_{i_0 \dots i_p}\})) = 0.$$

Since the higher direct images of the “smooth”  $E$ -valued  $(n, q)$ -forms are fine sheaves, there exists a  $(p-1)$ -cochain  $\{b_{i_0 \dots i_{p-1}}\}$  valued in the (not necessarily  $\bar{\partial}$ -closed)  $E$ -valued  $(n, q)$ -forms such that  $\{\mathcal{H}(\alpha_{i_0 \dots i_p})\} = \delta(\{b_{i_0 \dots i_{p-1}}\})$ . In fact, we can concretely construct  $\{b_{i_0 \dots i_{p-1}}\}$  by using a partition of unity associated to  $\mathcal{U}$  from  $\mathcal{H}(\alpha_{i_0 \dots i_p})$  (see Lemma 4.2). (The construction of  $\{b_{i_0 \dots i_{p-1}}\}$  plays an important role in the proof of Theorem 1.3.) Since  $\mathcal{H}(\alpha_{i_0 \dots i_p})$  is  $\bar{\partial}$ -closed, we can easily see that

$$\delta(\{\bar{\partial}b_{i_0 \dots i_{p-1}}\}) = \bar{\partial}\delta(\{b_{i_0 \dots i_{p-1}}\}) = \bar{\partial}\{\mathcal{H}(\alpha_{i_0 \dots i_p})\} = 0.$$

Then, by the same argument, there exists  $(p-2)$ -cochain  $\{b_{i_0 \dots i_{p-2}}\}$  valued in the  $E$ -valued  $(n, q+1)$ -forms such that  $\{\bar{\partial}b_{i_0 \dots i_{p-1}}\} = \delta(\{b_{i_0 \dots i_{p-2}}\})$ . Since  $\bar{\partial}b_{i_0 \dots i_{p-1}}$  is also  $\bar{\partial}$ -closed, we obtain  $\delta(\{\bar{\partial}b_{i_0 \dots i_{p-2}}\}) = 0$ . Therefore there exists  $(p-3)$ -cochain  $\{b_{i_0 \dots i_{p-3}}\}$  valued in the  $E$ -valued  $(n, q+2)$ -forms such that  $\{\bar{\partial}b_{i_0 \dots i_{p-2}}\} = \delta(\{b_{i_0 \dots i_{p-3}}\})$ . By repeating this process, for  $k = 1, 2, \dots, p$ , we can obtain  $(p-k)$ -cochain  $\{b_{i_0 \dots i_{p-k}}\}$  valued in the  $E$ -valued  $(n, q+k-1)$ -forms satisfying the following equalities:

$$(*) \left\{ \begin{array}{ll} \{\mathcal{H}(\alpha_{i_0 \dots i_p})\} &= \delta(\{b_{i_0 \dots i_{p-1}}\}), \\ \{\bar{\partial}b_{i_0 \dots i_{p-1}}\} &= \delta(\{b_{i_0 \dots i_{p-2}}\}), \\ &\vdots \\ \{\bar{\partial}b_{i_0 i_1}\} &= \delta(\{b_{i_0}\}). \end{array} \right.$$

By the last equality, we can easily check  $\delta(\{\bar{\partial}b_{i_0}\}) = 0$ , which says that  $\{\bar{\partial}b_{i_0}\}$  determines the  $E$ -valued  $(n, p+q)$ -form on  $X$ . We define  $\varphi_{p,q}(\{\alpha_{i_0 \dots i_p}\})$  by  $\varphi_{p,q}(\{\alpha_{i_0 \dots i_p}\}) := [\{\bar{\partial}b_{i_0}\}]$ .

At the end of this subsection, we prove that the map  $\varphi_{p,q}$  is well-defined. When  $p$  is zero, the equality  $\{\alpha_{i_0}\} \equiv \{\alpha'_{i_0}\}$  implies that  $\mathcal{H}(\alpha_{i_0}) = \mathcal{H}(\alpha'_{i_0})$  on  $U_{i_0}$ . Therefore  $\varphi_{0,q}$  is well-defined. In the case  $p > 0$ , for given  $\{\alpha_{i_0 \dots i_p}\}$  and  $\{\alpha'_{i_0 \dots i_p}\}$  such that  $\{\alpha_{i_0 \dots i_p}\} \equiv \{\alpha'_{i_0 \dots i_p}\} + \delta(\{c_{i_0 \dots i_{p-1}}\})$  for some  $(p-1)$ -cochain  $\{c_{i_0 \dots i_{p-1}}\}$ , we take an arbitrary  $\{b_{i_0 \dots i_{p-k}}\}$  satisfying  $(*)$  (resp.  $\{b'_{i_0 \dots i_{p-k}}\}$ ) satisfying  $(*)$  for  $\{\alpha_{i_0 \dots i_p}\}$  (resp.  $\{\alpha'_{i_0 \dots i_p}\}$ ). Then there exists a  $(p-2)$ -cochain  $\{d_{i_0 \dots i_{p-2}}\}$  such that

$$\{b_{i_0 \dots i_{p-1}} - b'_{i_0 \dots i_{p-1}} - \mathcal{H}(c_{i_0 \dots i_{p-1}})\} = \delta(\{d_{i_0 \dots i_{p-2}}\})$$

since the left hand side is  $\delta$ -closed. Putting  $c_{i_0 \dots i_{p-2}} = \bar{\partial}d_{i_0 \dots i_{p-2}}$ , we can take a  $(p-3)$ -cochain  $\{d_{i_0 \dots i_{p-3}}\}$  satisfying

$$\{b_{i_0 \dots i_{p-2}} - b'_{i_0 \dots i_{p-2}} - c_{i_0 \dots i_{p-2}}\} = \delta(\{d_{i_0 \dots i_{p-3}}\})$$

since the right hand side is  $\delta$ -closed. By putting  $c_{i_0 \dots i_{p-3}} = \bar{\partial}d_{i_0 \dots i_{p-3}}$  again, we can repeat this process, and thus we obtain  $c_{i_0 \dots i_{p-k}} := \bar{\partial}d_{i_0 \dots i_{p-k}}$  for  $k = 1, 2, \dots, p$ . Then, by the same argument, we can easily check  $\delta\{b_{i_0} - b'_{i_0} - c_{i_0}\} = 0$ , which says that  $\{b_{i_0} - b'_{i_0} - c_{i_0}\}$  determines the  $E$ -valued  $(n, p+q-1)$ -form  $\eta$  on  $X$ . Then we have  $\{\bar{\partial}b_{i_0}\} = \{\bar{\partial}b'_{i_0}\} + \bar{\partial}\eta$

since  $c_{i_0}$  is  $\bar{\partial}$ -closed. Therefore  $\{\bar{\partial}b_{i_0}\}$  and  $\{\bar{\partial}b'_{i_0}\}$  give the same cohomology class. We can easily see that the map  $\varphi_{p,q}$  does not depend on the choice of Stein open covers.

**3.3. Injectivity of  $\varphi_{p,q}$ .** In this subsection, we will show that the map  $\varphi_{p,q}$  constructed in subsection 3.2 is injective, by using the theory of harmonic integrals again.

First we consider the case  $p = 0$ . If we have  $\varphi_{0,q}(\{\alpha_{i_0}\}) \equiv 0$ , then there exists an  $E$ -valued  $(n, q-1)$ -form  $\eta$  on  $X$  such that  $\{\mathcal{H}(\alpha_{i_0})\} = \bar{\partial}\eta$ . Since the map

$$\mathcal{H}: H^q(V_{i_0}, K_X \otimes E) \xrightarrow{\cong} \mathcal{H}^{n,q}(V_{i_0}, E, \Phi_{U_{i_0}}).$$

is the inverse map of the natural quotient map (see Theorem 2.1), we have  $\alpha_{i_0} \equiv [\mathcal{H}(\alpha_{i_0})]$ . Therefore we obtain

$$\alpha_{i_0} \equiv [\mathcal{H}(\alpha_{i_0})] \equiv [\bar{\partial}\eta] \equiv 0.$$

From now on, we consider the case  $p > 0$ . We assume that  $\varphi_{p,q}(\{\alpha_{i_0\dots i_p}\}) \equiv 0$  for a  $p$ -cocycle  $\{\alpha_{i_0\dots i_p}\}$ . It is sufficient for the proof to construct a  $(p-1)$ -cochain  $\{c_{i_0\dots i_{p-1}}\}$  valued in the  $E$ -valued  $(n, q)$ -forms satisfying the following properties:

- $\{\alpha_{i_0\dots i_p}\} \equiv \delta(\{[c_{i_0\dots i_{p-1}}]\})$ .
- $c_{i_0\dots i_{p-1}}$  is  $\bar{\partial}$ -closed.

When we take a  $(p-k)$ -cochain  $\{b_{i_0\dots i_{p-k}}\}$  satisfying  $(*)$  for  $k = 1, 2, \dots, p$ , we have  $\varphi_{p,q}(\{\alpha_{i_0\dots i_p}\}) = [\{\bar{\partial}b_{i_0}\}] \equiv 0$ . Therefore there exists an  $E$ -valued  $(n, p+q-1)$ -form  $\eta$  on  $X$  such that  $\{\bar{\partial}b_{i_0}\} = \bar{\partial}\eta$ . By putting  $c_{i_0} := b_{i_0} - \eta$ , we have the following properties:

- $\{\bar{\partial}b_{i_0i_1}\} = \delta(\{c_{i_0}\})$ .
- $c_{i_0}$  is  $\bar{\partial}$ -closed.

Then we can obtain  $0 = \delta(\{\mathcal{H}(c_{i_0})\})$  since the left hand side in the above equality is  $\bar{\partial}$ -exact on  $V_{i_0i_1}$ . Further we can take an  $E$ -valued  $(n, p+q-2)$ -form  $d_{i_0}$  such that  $c_{i_0} = \mathcal{H}(c_{i_0}) + \bar{\partial}d_{i_0}$  since  $c_{i_0}$  and  $\mathcal{H}(c_{i_0})$  determine the same cohomology class. From the above argument, we can easily see that

$$\{\bar{\partial}b_{i_0i_1}\} = \delta(\{\mathcal{H}(c_{i_0}) + \bar{\partial}d_{i_0}\}) = \delta(\{\bar{\partial}d_{i_0}\}) = \bar{\partial}\delta(\{d_{i_0}\}).$$

Putting  $\{c_{i_0i_1}\} := \{b_{i_0i_1}\} - \delta(\{d_{i_0}\})$ , we have the following properties:

- $\{\bar{\partial}b_{i_0i_1i_2}\} = \delta(\{c_{i_0i_1}\})$ .
- $c_{i_0i_1}$  is  $\bar{\partial}$ -closed.

By repeating this process, we obtain a  $(p-1)$ -cochain  $\{c_{i_0\dots i_{p-1}}\}$  valued in the  $\bar{\partial}$ -closed  $E$ -valued  $(n, q)$ -forms satisfying  $\{\mathcal{H}(\alpha_{i_0\dots i_p})\} = \delta(\{c_{i_0\dots i_{p-1}}\})$ . Then we can easily see that

$$\{\alpha_{i_0\dots i_p}\} \equiv \{[\mathcal{H}(\alpha_{i_0\dots i_p})]\} \equiv \delta(\{[c_{i_0\dots i_{p-1}}]\}).$$

This completes the proof of the injectivity.

**3.4. On the image of  $\varphi_{p,q}$ .** In this subsection, we prove that  $\text{Im } \varphi_{p,q} \cap \text{Im } \varphi_{p',q'} = \{0\}$  when  $p \neq p'$  and  $p + q = p' + q'$ . Without loss of generality, we may assume  $p < p'$ . For a  $p$ -cocycle  $\{\alpha_{i_0 \dots i_p}\}$  and a  $p'$ -cocycle  $\{\alpha'_{i_0 \dots i_{p'}}\}$  valued in the (Dolbeault) cohomology group, we assume that  $\varphi_{p,q}(\{\alpha_{i_0 \dots i_p}\}) = \varphi_{p',q'}(\{\alpha'_{i_0 \dots i_{p'}}\})$ . We take a  $(p-k)$ -cochain  $\{b_{i_0 \dots i_{p-k}}\}$  for  $k = 1, 2, \dots, p$  (resp. a  $(p' - k')$ -cochain  $\{b'_{i_0 \dots i_{p'-k'}}\}$  for  $k' = 1, 2, \dots, p'$ ) satisfying (\*). By the assumption, there exists an  $E$ -valued  $(n, p+q-1)$ -form  $\eta$  on  $X$  such that  $\{\bar{\partial}b_{i_0}\} - \{\bar{\partial}b'_{i_0}\} = \bar{\partial}\eta$ . Then, by putting  $c_{i_0} := b_{i_0} - b'_{i_0} - \eta$ , we have the same properties as in subsection 3.3:

$$\bullet \quad \{\bar{\partial}(b_{i_0 i_1} - b'_{i_0 i_1})\} = \delta(\{c_{i_0}\}). \quad \bullet \quad c_{i_0} \text{ is } \bar{\partial}\text{-closed.}$$

By repeating the same argument as in subsection 3.3, we can construct a  $(p-1)$ -cochain  $\{c_{i_0 \dots i_{p-1}}\}$  valued in the  $\bar{\partial}$ -closed  $E$ -valued  $(n, q)$ -forms satisfying  $\{\mathcal{H}(\alpha_{i_0 \dots i_p}) - \bar{\partial}b'_{i_0 \dots i_{p'}}\} = \delta(\{c_{i_0 \dots i_{p-1}}\})$ . Here we used the assumption of  $p < p'$ . By taking  $\mathcal{H}(\bullet)$ , we obtain  $\{\mathcal{H}(\alpha_{i_0 \dots i_p})\} = \delta(\{\mathcal{H}(c_{i_0 \dots i_{p-1}})\})$ , and thus we can show

$$\{\alpha_{i_0 \dots i_p}\} \equiv \{[\mathcal{H}(\alpha_{i_0 \dots i_{p-1}})]\} \equiv \delta(\{[\mathcal{H}(c_{i_0 \dots i_{p-1}})]\}).$$

This completes the proof.

**3.5. Proof of Corollary 1.2.** In this subsection, we give a proof of Corollary 1.2. Let  $f: X \rightarrow Y$  be a proper surjective holomorphic map from a compact Kähler manifold  $X$  to an analytic space  $Y$ . Since  $E$  is assumed to admit a metric with semi-positive curvature in the sense of Nakano, we have

$$H^\ell(X, K_X \otimes E) \supset \bigoplus_{p+q=\ell} \text{Im } \varphi_{p,q}.$$

by Theorem 1.1. Takegoshi proved that the Leray spectral sequence for  $f$

$$H^p(Y, R^q f_*(K_X \otimes E)) \implies H^{p+q}(X, K_X \otimes E)$$

degenerates at  $E_2$ -term (see [Tak95, I Decomposition theorem]). In particular, when  $X$  is a compact Kähler manifold,  $\dim H^\ell(X, K_X \otimes E)$  is equal to  $\sum_{p+q=\ell} \dim H^p(Y, R^q f_*(K_X \otimes E))$ . This leads to the first conclusion of Corollary 1.2.

Now we prove the latter conclusion of Corollary 1.2. Let  $F$  be a line bundle on  $X$  admitting a metric with semi-positive curvature. The multiplication maps  $\Phi_s$  and  $\Psi_s$  induced by a section  $s$  of  $F^m$  satisfy the following commutative diagram:

$$\begin{array}{ccc} H^p(Y, R^q f_*(K_X \otimes E \otimes F)) & \xrightarrow{\varphi_{p,q}} & H^{p+q}(X, K_X \otimes E \otimes F) \\ \Phi_s \downarrow & & \downarrow \Psi_s \\ H^p(Y, R^q f_*(K_X \otimes E \otimes F^{m+1})) & \xrightarrow{\varphi'_{p,q}} & H^{p+q}(X, K_X \otimes E \otimes F^{m+1}) \end{array}$$

Indeed, for a given  $\{\alpha_{i_0 \dots i_p}\} \in H^p(Y, R^q f_*(K_X \otimes E \otimes F))$ , we can check  $[s\mathcal{H}(\alpha_{i_0 \dots i_p})] \equiv s\alpha_{i_0 \dots i_p}$ . Then we have  $s\mathcal{H}(\alpha_{i_0 \dots i_p}) \equiv \mathcal{H}(s\alpha_{i_0 \dots i_p})$  since  $s\mathcal{H}(\alpha_{i_0 \dots i_p})$  is also harmonic by [Tak95, Proposition 4.4]. It implies that, for a  $(p-k)$ -cochain  $\{b_{i_0 \dots i_{p-k}}\}$  satisfying (\*)



for  $\{\alpha_{i_0 \dots i_p}\}$ , the  $(p-k)$ -cochain  $\{sb_{i_0 \dots i_{p-k}}\}$  also satisfies  $(*)$  for  $\{s\alpha_{i_0 \dots i_p}\}$ . Therefore the above diagram is commutative.

Since  $F$  admits a metric with semi-positive curvature, the vector bundles  $E \otimes F$  and  $E \otimes F^{m+1}$  are so. The map  $\varphi_{p,q}$  and  $\varphi'_{p,q}$  are injective by Theorem 1.1, and further  $\Psi_s$  is also injective by the standard injectivity theorem (see [Eno90], [Tak95]). Therefore the multiplication map  $\Phi_s$  is also injective.

#### 4. VANISHING THEOREM FOR COHOMOLOGY GROUPS OF HIGHER DIRECT IMAGES

In this section, we give a proof of Theorem 1.3. The proof is a combination of Theorem 1.1 and techniques to solve the  $\bar{\partial}$ -equation with the  $L^2$ -estimate.

**Theorem 4.1** (= Theorem 1.3). *Let  $f: X \rightarrow Y$  be a proper surjective holomorphic map from a weakly pseudoconvex Kähler manifold  $X$  to an analytic space  $Y$  and  $E$  be a vector bundle on  $X$ . If  $E$  admits a metric  $g$  satisfying  $\sqrt{-1}\Theta_g(E) \geq f^*\sigma \otimes \text{id}_E$  for some Kähler form  $\sigma$  on  $Y$  in the sense of Nakano, then we have*

$$H^p(Y, R^q f_*(K_X \otimes E)) = 0 \quad \text{for any } p > 0 \text{ and } q \geq 0.$$

A Kähler form  $\sigma$  on the regular locus  $Y_{\text{reg}}$  is said to be a Kähler form on  $Y$  if for every point  $y \in Y$  there exist a local embedding  $i: U_y \hookrightarrow \mathbb{C}^m$  of an open neighborhood  $U_y$  of  $y$  and a Kähler form  $\tilde{\sigma}$  on a neighborhood of  $i(U_y)$  such that  $\sigma = i^*\tilde{\sigma}$  on  $U_y \cap Y_{\text{reg}}$ . The pull-back  $f^*\sigma$  is the extension of  $f^*\sigma$  on  $f^{-1}(Y_{\text{reg}})$  to  $X$ , which is a semi-positive  $(1,1)$ -form on  $X$ . We show that the natural map  $\varphi_{p,q}$  is the zero map under the assumption on the curvature of  $E$ . Then we obtain a vanishing theorem for the higher cohomology groups of  $R^q f_*(K_X \otimes E)$  by Theorem 1.1.

In the same way as in Section 3, we fix a Stein open cover  $\mathcal{U} := \{U_i\}_{i \in I}$  of  $Y$ , and compute the Čech cohomology group. It is easy to prove the following lemma, which gives an explicit form of  $b_{i_0 \dots i_{p-k}}$  satisfying  $(*)$ . The explicit form plays an important role when we solve the  $\bar{\partial}$ -equation. From now on, we fix a partition of unity  $\{\rho_i\}_{i \in I}$  associated to  $\mathcal{U}$ , and put  $\tilde{\rho}_i := f^*\rho_i$ .

**Lemma 4.2.** *Let  $\{x_{i_0 \dots i_\ell}\}$  be an  $\ell$ -cocycle valued in the  $E$ -valued  $(n, q)$ -forms. Then  $y_{i_0 \dots i_{\ell-1}} := \sum_{j \in I} \tilde{\rho}_j x_{ji_0 \dots i_{\ell-1}}$  satisfies*

$$\{x_{i_0 \dots i_\ell}\} = \delta(\{y_{i_0 \dots i_{\ell-1}}\}).$$

The proof is a straightforward computation, and thus we omit it. Assume that  $p > 0$ . For an arbitrary class in  $\check{H}^p(\mathcal{U}, R^q f_*(K_X \otimes E))$ , we take a  $p$ -cocycle  $\{\alpha_{i_0 \dots i_p}\}$  representing the class. By the above lemma, the  $E$ -valued form  $b_{i_0 \dots i_{p-k}}$  defined inductively by

$$b_{i_0 \dots i_{p-1}} := \sum_{j \in I} \tilde{\rho}_j \mathcal{H}(\alpha_{ji_0 \dots i_{p-1}}) \quad \text{and} \quad b_{i_0 \dots i_{p-k}} := \sum_{j \in I} \tilde{\rho}_j \bar{\partial} b_{ji_0 \dots i_{p-k}}$$

satisfies  $(*)$ . In particular, we have

$$\bar{\partial} b_{i_0} = \bar{\partial} \left( \sum_{j \in I} \tilde{\rho}_j \bar{\partial} b_{ji_0} \right) = \sum_{j \in I} \bar{\partial} \tilde{\rho}_j \wedge \bar{\partial} b_{ji_0} \quad \text{on } U_{i_0}.$$

From the above equality, we will prove that the  $E$ -valued  $(n, p+q)$ -form  $\varphi_{p,q}(\{\alpha_{i_0 \dots i_p}\}) = \{\bar{\partial} b_{i_0}\}$  is  $\bar{\partial}$ -exact on  $X$ , namely there exists an  $E$ -valued  $(n, p+q-1)$ -form  $\eta$  on  $X$  such that  $\{\bar{\partial} b_{i_0}\} = \bar{\partial}\eta$ . It implies that the map  $\varphi_{p,q}$  is the zero map.

Fix a complete Kähler form  $\omega$  on  $X$ . Note that  $X$  admits a complete Kähler form since  $X$  is a weakly pseudoconvex Kähler manifold. We take a metric  $g$  on  $E$  such that  $\sqrt{-1}\Theta_g(E) \geq f^*\sigma \otimes \text{id}_E$  holds for some Kähler form  $\sigma$  in the sense of Nakano. Further we take an exhaustive smooth plurisubharmonic function  $\Phi$  on  $X$ . To solve the  $\bar{\partial}$ -equation  $\{\bar{\partial} b_{i_0}\} = \bar{\partial}\eta$  by the standard technique (cf. [Dem82], [Ohs84]), we consider the new metric  $h$  on  $E$  defined by

$$h := ge^{-2\chi \circ \Phi}.$$

Here  $\chi: \mathbb{R} \rightarrow \mathbb{R}$  is an increasing convex function, which will be suitably chosen later. The composite function  $\chi \circ \Phi$  is also plurisubharmonic, and thus we have

$$\sqrt{-1}\Theta_h(E) = \sqrt{-1}\Theta_g(E) + (2\sqrt{-1}\partial\bar{\partial}\chi \circ \Phi) \otimes \text{Id}_E \geq_{\text{Nak}} f^*\sigma \otimes \text{id}_E.$$

From now on, we mainly handle the norm with respect to  $h$  (not  $g$ ) and  $\omega$ . (We often omit the subscript.) We consider the linear functional

$$\text{Im } \bar{\partial}^* \longrightarrow \mathbb{C} \quad \text{defined by} \quad w = \bar{\partial}^* v \longmapsto \langle\langle v, \{\bar{\partial} b_{i_0}\} \rangle\rangle_{h,\omega},$$

where  $\text{Im } \bar{\partial}^*$  is the range of the closed operator  $\bar{\partial}^*$  in the  $L^2$ -space  $L_{(2)}^{n,p+q-1}(X, E)_{h,\omega}$ . For the proof, it is sufficient to prove that the above linear map is (well-defined and) bounded, that is, there exists a positive constant  $C$  such that

$$|\langle\langle v, \{\bar{\partial} b_{i_0}\} \rangle\rangle|^2 \leq C \|\bar{\partial}^* v\|^2 \quad \text{for any } v \in \text{Dom } \bar{\partial}^*.$$

Indeed, we can obtain an  $E$ -valued  $(n, p+q-1)$ -form  $\eta$  on  $X$  such that  $\langle\langle v, \{\bar{\partial} b_{i_0}\} \rangle\rangle = \langle\langle \bar{\partial}^* v, \eta \rangle\rangle$  for any  $v \in \text{Dom } \bar{\partial}^*$  and  $\|\eta\|^2 \leq C$ , by the Hahn-Banach theorem and the Riesz representation theorem. It gives a solution of the  $\bar{\partial}$ -equation  $\{\bar{\partial} b_{i_0}\} = \bar{\partial}\eta$ .

It is sufficient for the above estimate to prove that there exists a positive constant  $C > 0$  such that

$$|\langle\langle v, \{\bar{\partial} b_{i_0}\} \rangle\rangle|^2 \leq C(\|\bar{\partial} v\|^2 + \|\bar{\partial}^* v\|^2)$$

for compactly supported smooth  $v$ . From this inequality, we know that the above inequality also holds for arbitrary  $v \in \text{Dom } \bar{\partial} \cap \text{Dom } \bar{\partial}^*$ . This is because, for a given  $v \in \text{Dom } \bar{\partial} \cap \text{Dom } \bar{\partial}^*$ , we can take a compactly supported smooth  $v_k$  such that  $v_k \rightarrow v$ ,  $\bar{\partial} v_k \rightarrow \bar{\partial} v$ , and  $\bar{\partial}^* v_k \rightarrow \bar{\partial}^* v$  in the  $L^2$ -space  $L_{(2)}^{n,\bullet}(X, E)_{h,\omega}$  by the density lemma (for example, see [Dem-book, (3.2) Theorem, Chapter VIII]). Here we used the condition that  $\omega$  is complete. Then, for any  $v \in \text{Dom } \bar{\partial}^*$ , we can easily see that

$$|\langle\langle v, \{\bar{\partial} b_{i_0}\} \rangle\rangle|^2 = |\langle\langle v_1, \{\bar{\partial} b_{i_0}\} \rangle\rangle|^2 \leq C \|\bar{\partial}^* v_1\|^2 \leq C \|\bar{\partial}^* v\|^2$$

by the orthogonal decomposition

$$v = v_1 + v_2 \in \text{Ker } \bar{\partial} \oplus (\text{Ker } \bar{\partial})^\perp.$$

To prove the above inequality for compactly supported smooth  $v$ , we consider the operator  $B_\delta := \omega_\delta \Lambda_\omega$  acting on the  $L^2$ -space  $L_{(2)}^{n,\bullet}(X, E)_{h,\omega}$ , where  $\omega_\delta$  is the Kähler form on  $X$  defined by  $\omega_\delta = \delta\omega + f^*\sigma$  and  $\Lambda_\omega$  is the (point-wise) adjoint operator of the wedge product  $\omega \wedge \bullet$  with respect to  $\langle \bullet, \bullet \rangle_{h,\omega}$ . The operator  $B_\delta$  is positive definite (for example see [Dem-book, (5.8) Proposition, Chapter VI]). In particular, the operator  $B_\delta$  has the inverse operator  $B_\delta^{-1}$ . By the Cauchy-Schwarz inequality, we have

$$|\langle v, \{\bar{\partial}b_{i_0}\} \rangle|^2 \leq \langle B_\delta^{-1} \{\bar{\partial}b_{i_0}\}, \{\bar{\partial}b_{i_0}\} \rangle \langle B_\delta v, v \rangle.$$

We first consider the limit of  $\langle B_\delta v, v \rangle$  when  $\delta$  goes to zero. By Bochner-Kodaira-Nakano identity and the assumption of  $\sqrt{-1}\Theta_h(E) \geq_{\text{Nak}} f^*\sigma \otimes \text{id}_E$ , we have

$$\begin{aligned} \lim_{\delta \rightarrow 0} \langle B_\delta v, v \rangle &= \langle f^*\sigma \Lambda_\omega v, v \rangle \\ &\leq \langle \sqrt{-1}\Theta_h(E) \Lambda_\omega v, v \rangle \\ &\leq \|\bar{\partial}v\|^2 + \|\bar{\partial}^*v\|^2. \end{aligned}$$

Now we prove that the norm  $\langle B_\delta^{-1} \{\bar{\partial}b_{i_0}\}, \{\bar{\partial}b_{i_0}\} \rangle_{X,h,\omega}$  on  $X$  is uniformly bounded with respect to  $\delta > 0$  if we suitably choose an increasing convex function  $\chi$ . For an arbitrary relatively compact set  $K \Subset X$ , we consider the norm  $\langle B_\delta^{-1} \{\bar{\partial}b_{i_0}\}, \{\bar{\partial}b_{i_0}\} \rangle_{K,h,\omega}$  on  $K$ . We can obtain

$$\begin{aligned} \langle B_\delta^{-1} \{\bar{\partial}b_{i_0}\}, \{\bar{\partial}b_{i_0}\} \rangle_{K,h,\omega} &\leq \sum_{\substack{i_0 \in I \text{ with} \\ U_{i_0} \cap K \neq \emptyset}} \langle B_\delta^{-1} \bar{\partial}b_{i_0}, \bar{\partial}b_{i_0} \rangle_{B_{i_0},h,\omega} \\ &\leq \sum_{\substack{i_0 \in I \text{ with} \\ U_{i_0} \cap K \neq \emptyset}} \sum_{\substack{j \in I \text{ with} \\ U_j \cap U_{i_0} \neq \emptyset}} \langle B_\delta^{-1} \bar{\partial}f^*\rho_j \wedge \bar{\partial}b_{ji_0}, \bar{\partial}f^*\rho_j \wedge \bar{\partial}b_{ji_0} \rangle_{B_{i_0},h,\omega} \\ &\leq D \sum_{\substack{i_0 \in I \text{ with} \\ U_{i_0} \cap K \neq \emptyset}} \sum_{\substack{j \in I \text{ with} \\ U_j \cap U_{i_0} \neq \emptyset}} \int_{B_{i_0}} |b_{ji_0}|_{h,\omega}^2 |\bar{\partial}\rho_j|_\sigma^2 dV_\omega \end{aligned}$$

from Lemma 4.2 and 4.3. The proof of Lemma 4.3 is given at the end of this section. The right hand side does not depend on  $\delta$  since the constant  $D$  in Lemma 4.3 is independent of  $\delta$ . Moreover we may assume that it becomes finite if we choose rapidly increasing function  $\chi$ . Therefore we obtain

$$|\langle v, \{\bar{\partial}b_{i_0}\} \rangle|^2 \leq C(\|\bar{\partial}v\|^2 + \|\bar{\partial}^*v\|^2)$$

for compactly supported smooth  $v$ . This completes the proof.

It remains to prove the following lemma:

**Lemma 4.3.** *Let  $\varphi$  be an  $E$ -valued  $(n, k)$ -form on  $X$  and  $\rho$  is a smooth function on  $Y$ . Then there exists a positive constant  $D$  (depending only on  $k$ ,  $\text{rank } E$ ) such that*

$$\langle B_\delta^{-1}(\bar{\partial}f^*\rho \wedge \varphi), (\bar{\partial}f^*\rho \wedge \varphi) \rangle_{h,\omega} \leq D |\varphi|_{h,\omega}^2 |\bar{\partial}\rho|_\sigma^2.$$

*Proof.* For a given point  $x \in X$ , we choose a local coordinate  $z := (z_1, z_2, \dots, z_n)$  centered at  $x$  such that

$$\omega = \frac{\sqrt{-1}}{2} \sum_{j=1}^n dz_j \wedge d\bar{z}_j \quad \text{and} \quad f^* \sigma = \frac{\sqrt{-1}}{2} \sum_{j=1}^n \lambda_j dz_j \wedge d\bar{z}_j \quad \text{at } x.$$

By taking a local embedding  $i: U_y \hookrightarrow \mathbb{C}^m$  of a neighborhood  $U_y$  of  $y := f(x) \in Y$ , we can assume that  $Y$  is an open set in  $\mathbb{C}^m$  since  $\rho$  (resp.  $\sigma$ ) can be seen as the pull-back of a smooth function (resp. a Kähler form) on  $\mathbb{C}^m$ . We choose a local coordinate  $t := (t_1, t_2, \dots, t_m)$  centered at  $y$  such that

$$\sigma = \frac{\sqrt{-1}}{2} \sum_{j=1}^m dt_j \wedge d\bar{t}_j \quad \text{at } y = f(x).$$

When we write  $f = (f_1, f_2, \dots, f_m)$  with respect to these coordinates, we have

$$\sum_{p=1}^m \frac{\partial f_p}{\partial z_k} \frac{\partial \overline{f_p}}{\partial z_\ell} = \delta_{k\ell} \lambda_k \quad \text{at } x.$$

For the local expressions with respect to these coordinates

$$\varphi = \sum_I \sum_{i=1}^{\text{rank } E} \varphi_{I,i}(z) e_i \otimes dz \wedge d\bar{z}_I \quad \text{and} \quad \bar{\partial} f^* \rho = \sum_{k=1}^n \sum_{j=1}^m \frac{\partial \rho}{\partial \bar{t}_j}(f(z)) \frac{\partial \overline{f_j}}{\partial z_k}(z) d\bar{z}_k,$$

we can easily see that

$$|\varphi|_\omega^2 = \sum_I \sum_{i=1}^{\text{rank } E} |\varphi_{I,i}|^2 \quad \text{at } x \quad \text{and} \quad |\bar{\partial} \rho|_\sigma^2 = \sum_{j=1}^m \left| \frac{\partial \rho}{\partial \bar{t}_j}(f(0)) \right|^2 \quad \text{at } y.$$

Here  $dz$  is  $dz := dz_1 \wedge dz_2 \cdots \wedge dz_n$ ,  $d\bar{z}_I$  is  $d\bar{z}_I := d\bar{z}_{i_1} \wedge d\bar{z}_{i_2} \cdots \wedge d\bar{z}_{i_k}$  for an ordered multi-index  $I = \{i_1 < i_2 < \cdots < i_k\}$ , and  $e_i$  is a local frame of  $E$  that gives an orthonormal basis at  $x$ . Then, by putting

$$g_k(z) := \sum_{j=1}^m \frac{\partial \rho}{\partial \bar{t}_j}(f(z)) \frac{\partial \overline{f_j}}{\partial z_k}(z),$$

we have

$$|g_k(0)|^2 \leq |\bar{\partial} \rho|_\sigma^2 \left( \sum_{p=1}^m \left| \frac{\partial f_p}{\partial z_k}(0) \right|^2 \right)$$

by the Cauchy-Schwarz inequality. Let  $\mu_j$  be an eigenvalue of  $\omega_\delta$  with respect to  $\omega$ . We remark that  $\mu_j = \delta + \lambda_j$  at  $x$ . Then, by straightforward computations, we obtain

$$\begin{aligned}
& \langle B_\delta^{-1}(\bar{\partial}f^*\rho \wedge \varphi), (\bar{\partial}f^*\rho \wedge \varphi) \rangle_{h,\omega} \\
&= \sum_J \sum_{i=1}^{\text{rank } E} \left( \sum_{I \cup \{k\} = J} \text{sgn} \binom{J}{Ik} \varphi_{I,i} g_k(0) \right) \left( \sum_{I' \cup \{k'\} = J} \text{sgn} \binom{J}{I'k'} \varphi_{I',i} g_{k'}(0) \right) \left( \sum_{j \in J} \mu_j \right)^{-1} \\
&\leq C_1 \sum_J \sum_{i=1}^{\text{rank } E} \sum_{I \cup \{k\} = J} |\varphi_{I,i}|^2 |g_k(0)|^2 \left( \sum_{j \in J} \mu_j \right)^{-1} \\
&\leq C_2 |\varphi|_\omega^2 |\bar{\partial}\rho|_\sigma^2 \sum_J \frac{\sum_{k \in J} \sum_{p=1}^m \left| \frac{\partial f_p}{\partial z_k}(0) \right|^2}{\sum_{j \in J} (\delta + \sum_{p=1}^m \left| \frac{\partial f_p}{\partial z_j}(0) \right|^2)}
\end{aligned}$$

for some constants  $C_1, C_2 > 0$ . The first inequality follows from the fundamental inequality  $(\sum_{i=1}^N |a_i|)^2 \leq 2^{N-1} \sum_{i=1}^N |a_i|^2$  and the second inequality follows from the above estimate for  $g_k(0)$ . The last term is smaller than or equal to one for any  $\delta > 0$ . This completes the proof.  $\square$

*Remark 4.4.* (1) In his paper [Ohs84], Ohsawa proved Theorem 1.3 in the case  $q = 0$ , whose strategy is as follows: He showed that, for a  $\bar{\partial}$ -closed  $E$ -valued  $(n, k)$ -form  $f$  on a complete Kähler manifold, there exists a solution  $g$  of the  $\bar{\partial}$ -equation  $\bar{\partial}g = f$  satisfying  $\|g\|_\sigma \leq C\|f\|_\sigma$  for some positive constant  $C$  if  $\|f\|_\sigma < \infty$ . Here  $\|\bullet\|_\sigma$  denotes the  $L^2$ -norm define by  $\|\bullet\|_\sigma := \lim_{\delta \rightarrow 0} \|\bullet\|_{\delta\omega + \sigma}$ . (See [Ohs84, Theorem 2.8] for the precise statement.) From this celebrated result, he obtained Theorem 1.3 by inductively constructing solutions of the  $\bar{\partial}$ -equation.

(2) However, we can not expect  $\|\bar{\partial}b_{i_0}\|_\sigma < \infty$  in the case  $q > 0$ . In fact, we can prove that  $\|\bar{\partial}b_{i_0}\|_\sigma = \infty$  in this case, by using [Tak95, Theorem 5.2]. For this reason, we estimate the limit of  $\langle B_\delta^{-1} \{\bar{\partial}b_{i_0}\}, \{\bar{\partial}b_{i_0}\} \rangle$  instead of  $\|\bar{\partial}b_{i_0}\|_\sigma$ .

(3) By the same computation as the proof of Lemma 4.3, we can check that  $\lim_{\delta \rightarrow 0} \langle B_\delta^{-(p+q)} \{\bar{\partial}b_{i_0}\}, \{\bar{\partial}b_{i_0}\} \rangle$  is finite. If we can construct a solution  $g$  of the  $\bar{\partial}$ -equation  $\bar{\partial}g = f$  such that  $\lim_{\delta \rightarrow 0} \langle B_\delta^{-(k-1)} g, g \rangle < \infty$  from the assumption of  $\lim_{\delta \rightarrow 0} \langle B_\delta^{-k} f, f \rangle < \infty$ , we can prove Theorem 1.3 by the same method as in [Ohs84]. However this strategy did not succeed, and thus we need the injective map  $\varphi_{p,q}$  constructed in Theorem 1.1.

## REFERENCES

- [Dem-book] J.-P. Demailly. *Complex analytic and differential geometry*. Lecture Notes on the web page of the author.
- [Dem82] J.-P. Demailly. *Estimations  $L^2$  pour l'opérateur  $\bar{\partial}$  d'un fibré vectoriel holomorphe semi-positif au-dessus d'une variété kählérienne complète*. Ann. Sci. École Norm. Sup(4). **15** (1982), no. 3, 457–511.
- [Eno90] I. Enoki. *Kawamata-Viehweg vanishing theorem for compact Kähler manifolds*. Einstein metrics and Yang-Mills connections (Sanda, 1990), 59–68.
- [EV92] H. Esnault, E. Viehweg. *Lectures on vanishing theorems*. DMV Seminar, **20**. Birkhäuser Verlag, Basel, (1992).
- [Fn15] O. Fujino. *On semipositivity, injectivity, and vanishing theorems*. Preprint, a survey for Zucker 65, arXiv:1503.06503v3.
- [Fs15] T. Fujisawa. *A remark on the decomposition theorem for direct images of canonical sheaves tensorized with semipositive vector bundles*. Preprint, arXiv:1512.03887v1.
- [Kol86a] J. Kollár. *Higher direct images of dualizing sheaves. I*. Ann. of Math. (2) **123** (1986), no. 1, 11–42.
- [Kol86b] J. Kollár. *Higher direct images of dualizing sheaves. II*. Ann. of Math. (2) **124** (1986), no. 1, 171–202.
- [Mat15] S. Matsumura. *Injectivity theorems with multiplier ideal sheaves and their applications*. Complex Analysis and Geometry, **144** of the series Springer Proceedings in Mathematics & Statistics, 241–255
- [Ohs84] T. Ohsawa. *Vanishing theorems on complete Kähler manifolds*. Publ. Res. Inst. Math. Sci. **20** (1984), no. 1, 21–38.
- [Sko78] H. Skoda. *Morphismes surjectifs de fibrés vectoriels semi-positifs*. Ann. Sci. École Norm. Sup. (4) **11** (1978), no. 4, 577–611.
- [Tak95] K. Takegoshi. *Higher direct images of canonical sheaves tensorized with semi-positive vector bundles by proper Kähler morphisms*. Math. Ann. **303** (1995), no. 3, 389–416.

MATHEMATICAL INSTITUTE, TOHOKU UNIVERSITY, 6-3, ARAMAKI AZA-AOBA, AOBA-KU, SENDAI 980-8578, JAPAN.

*E-mail address*: mshinichi@m.tohoku.ac.jp, mshinichi0@gmail.com